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Yangian $Y(sl(2))$ in Hydrogen Atom

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The Yangian $Y(sl(2))$ is studied in the usual hydrogen atom. Its generators are expressed in terms of the angular momentum operators and the so-called Runge–Lenz vector. The energy is found to play the role of the deformation parameter. At the critical point from the bound state to the free state, the Yangian $Y(sl(2))$ reduces to the loop algebra $L(sl(2))$. The corresponding matrix elements of the $Y(sl(2))$ generators are also discussed for the energy eigenstates.

The Yangian is an interesting problem in mathematical physics. It was first introduced by Drinfeld (1985, 1987). Scientists have found there is Yangian symmetry in many physical models, such as long-range interaction models (Haldane *et al.*, 1992; Kato and Kuramoto, 1995; Hikami, 1995), the one-dimensional Hubbard model (Uglov and Korepin, 1984; Gohmann and Inozemtsev, 1996), the Heisenberg model (Bernard, 1993), and the two-dimensional chiral model with or without topological terms (Bardeen, 1991; Schoutens, 1994). It is also significant that Yangian may play an important role in exploring the multiparton amplitudes in QCD (Ge, 1996; Bardeen, 1996).

In this paper, the Yangian $Y(sl(2))$ will be studied in the usual hydrogen atom from the point of view of quantum mechanics. It is well known that there is the $SO(4)$ symmetry in the hydrogen atom (Schiff, 1968), and generators of $SO(4)$ are realized in terms of the angular momentum operator $\mathbf{L}$ and so-called Runge–Lenz vector $\mathbf{R}$ for fixed energy level. The q-hydrogen atom (Kibler and Negadi, 1991; Zhang, 1995) has been studied by deforming the corresponding symmetry. However, the set $\{\mathbf{L}, \mathbf{R}\}$ becomes unclosed to the commutative product when the energy is not fixed. It is interesting to study the algebraic structure in the hydrogen atom.

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Inspired by the Yangian, we will investigate the algebra in the hydrogen atom. It will be seen that {**L**, **R**} can also form a Yangian $Y(sl(2))$. We shall give the generators of $Y(sl(2))$, which are expressed by {**L**, **R**}. The commutator brackets of the $Y(sl(2))$ generators will be calculated for our realization. These generators are shown to obey the commutation relation. The properties of the generators are also discussed.

Let us start from the hydrogen atom with the well-known Hamiltonian

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{|\mathbf{r}|} \tag{1}$$

where μ and e stand for the mass and electric charge of the electron, respectively. It has been pointed out (Schiff, 1968) that besides the angular momentum **L**, there is another conserved quantity, the Runge–Lenz vector **R**, and then satisfy

$$[\mathbf{L}, H] = [\mathbf{R}, H] = 0 \tag{2}$$

and

$$\mathbf{R} = \frac{1}{2\mu e^2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r} \tag{3}$$

According to Drinfeld (1985, 1987), the Yangian $Y(sl(2))$ is generated by the generators $\{I_\pm, I_3\}$ and $\{J_\pm, J_3\}$ with the commutation relation

$$\begin{gathered}[I_3, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = 2I_3 \\ [I_3, J_\pm] = [J_3, I_\pm] = \pm J_\pm, \qquad [I_\pm, J_\mp] = \pm 2J_3 \\ [I_3, J_3] = [I_\pm, J_\pm] = 0\end{gathered} \tag{4}$$

and

$$\begin{gathered}[J_3, [J_+, J_-]] = \frac{\alpha^2}{4} I_3(J_-I_+ - I_-J_+) \\ [J_\pm, [J_3, J_\pm]] = \frac{\alpha^2}{4} I_\pm(J_\pm I_3 - I_\pm J_3) \\ 2[J_3, [J_3, J_\pm]] \pm [J_\pm, [J_\pm, J_\mp]] = \frac{\alpha^2}{2}\{I_3(J_\pm I_3 - I_\pm J_3) + I_\pm(I_-J_+ - J_-I_+)\}\end{gathered} \tag{5}$$

where α is the deformation parameter.

From (4), one easily finds that I_a obeys the same commutation relations as L_a. Therefore, we can directly set

$$I_3 = L_3, \qquad I_\pm = L_\pm = L_1 \pm iL_2 \tag{6}$$

In order to realize $Y(sl(2))$, we must determine the generators J_3 and $J_\pm$. Since the commutation relations of J_3 and $J_\pm$ with I_3 and $I_\pm$ are similar to the commutation relations of I_3 and $I_\pm$, one may suspect that $\{J_3, J_\pm\}$ should consist of $\{L_3, L_\pm\}$. It can be verified that Runge–Lenze vector satisfies

$$[L_3, R_\pm] = [R_3, L_\pm] = R_\pm$$
$$[L_3, R_3] = [L_\pm, R_\pm] = 0, \qquad [L_\pm, R_\mp] = \pm 2R_3 \tag{7}$$

in which we have

$$R_\pm = R_1 \pm iR_2$$

Therefore, the generators $\{J_3, J_\pm\}$ may also include the Runge–Lenz vector. By analyzing equations (5), (7) and the commutation relation of angular momentum, we find that generators $\{J_3, J_\pm\}$ can be chosen as

$$J_3 = \frac{1}{2}(L_+R_- - R_+L_-)$$
$$J_\pm = \pm(L_3R_\pm - R_3L_\pm) \tag{8}$$

Since I_a and R_a are conserved for the hydrogen atom (1), i.e., they obey the commutation relation (2), one easily sees that the generators $\{J_3, J_\pm\}$ are also conserved. They satisfy

$$[H, J_3] = [H, J_\pm] = 0 \tag{9}$$

On the other hand, the generators (6) and (8) can be verified to obey the commutation relation of $Y(sl(2))$, i.e., equations (4) and (5). Because of the definition (6), in which L_a $(a = 1, 2, 3)$ are usual angular momenta, the generators I_3 and $I_\pm$ must satisfy the first two formulas of (4) obviously. Using (6)–(8), one directly has $[I_3, J_\pm] = \pm J_\pm$. The other formulas of (4) can be verified similarly. Now, let us consider the commutation relations of the generators **J**. After tedious calculation, by making use of (7), (8), and

$$[R_a, R_b] = -\frac{i2}{\mu e^4} H\epsilon_{abc}L_c$$

we have

$$[J_+, J_-] = \frac{4H}{\mu e^4} I_3(2\mathbf{I}^2 + 1) + 2I_3 \tag{10}$$

Since $[I_3, J_3] = 0$ and

$$[\mathbf{I}^2, J_3] = J_-I_+ - I_-J_+$$

it can be obtained that

$$[J_3, [J_+, J_-]] = \frac{8H}{\mu e^4} I_3(J_-\mathrm{I}_+ - \mathrm{I}_-\mathrm{J}_+) \tag{11}$$

Comparing with the first equation of (5), we see that the generators (6) and (8) indeed satisfy the commutation relations of $Y(sl(2))$. In equation (11), the Hamiltonian H can be looked upon as a constant. This is due to its commuting with all generators of $Y(sl(2))$ in our discussion. Similarly, we also determine the other commutation relations of the generators $\mathbf{J}$ as follows:

$$[J_3, J_\pm] = \pm\frac{2H}{\mu e^4} I_\pm(2\mathbf{I}^2 + 1) \pm I_\pm \tag{12}$$

and we obtain

$$[J_\pm, [J_3, J_\pm]] = \frac{8H}{\mu e^4} I_\pm(J_\pm I_3 - I_\pm J_3) \tag{13}$$

In deriving (13), we applied equation (4) and

$$[\mathbf{I}^2, J_\pm] = 2(J_\pm I_3 - I_\pm J_3)$$

It can been seen that the commutation relation (I^3) corresponds to the second equation of (5). If we use the commutation relations (10) and (12), the last commutation relation of $Y(sl(2))$ can be determined as follows:

$$2[J_3, [J_3, J_\pm]] + [J_\pm, [J_\pm, J_\mp]] = \frac{4H}{\mu e^2} \{I_3(J_\pm I_3 - I_\pm J_3) + I_\pm(I_-J_+ - J_-I_+)\} \tag{14}$$

From (5), (11), (13), and (14), we see that the energy plays the role of the deformation parameter for fixed energy in our realization (6) and (8). When $E_n \to 0$, i.e., $n \to \infty$, the Yangian $Y(sl(2))$ becomes the loop algebra $L(sl(2))$. Therefore, the algebraic structure at the energy critical point of the transition of the hydrogen atom from the bound to the free states differs from that in the other case.

It is well known that the spherical harmonics $Y_{lm}(\theta, \phi) = |lm\rangle$ are the eigenstates of L_z and $\mathbf{L}^2$. From equation (6), we have

$$I_3|lm\rangle = m|lm\rangle$$
$$I_\pm|lm\rangle = [(l \mp m)(l \pm m + 1)]^{1/2}|lm \pm 1\rangle \tag{15}$$

Furthermore, using (4), we can derive

$$\langle l'm'|J_\pm|lm\rangle = \delta_{m'm\pm1}\langle l'm \pm 1|J_\pm|lm\rangle \tag{16}$$

and

$$\langle l'm'|J_3|lm\rangle = \delta_{m'm}\langle l'm|J_3|lm\rangle \tag{17}$$

respectively. The above two formulas just indicate the selection rules of he generators $\mathbf{J}$. From the commutation relation $[I_+, J_+] = [I_-J_-] = 0$, we can determine the recurrence relations of the matrix elements of generators $\mathbf{J}$ as

$$\langle l'm+1|J_+|lm\rangle = \sqrt{\frac{(l'-m)(l'+m+1)}{(l+m)(l-m+1)}}\langle l'm|J_+|lm-1\rangle$$

$$\langle l'm-1|J_-|lm\rangle = \sqrt{\frac{(l'+m)(l'-m+1)}{(l-m)(l+m+1)}}\langle l'm|J_-|lm+1\rangle$$

Making use of $[I_+, J_-] = [J_+, I_-] = 2J_3$, we also obtain

$$\begin{aligned} 2\langle l'm|J_3|lm\rangle &= \sqrt{(l'+m)(l'-m+1)}\langle l'm-1|J_-|lm\rangle \\ &\quad -\sqrt{(l-m)(l+m+1)}\langle l'm|J_-|lm+1\rangle \\ &= \sqrt{(l+m)(l-m+1)}\langle l'm|J_+|lm-1\rangle \\ &\quad -\sqrt{(l'-m)(l'+m+1)}\langle l'm+1|J_+|lm\rangle \end{aligned}$$

Equation (15) shows that the generators $\mathbf{I}$ represent the transition in the same quantum number l. Equations (16) and (17) indicate that the generators $\mathbf{J}$ may play the role of transition between states with different angular quantum number l, which selection rule is similar to the generators $\mathbf{I}$ for the quantum number m. The above conclusion is universal for $Y(sl(2))$, not confined to the realization of (6) and (8).

The Hilbert space of the hydrogen atom is spanned by the wave function

$$|nlm\rangle = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$R_{nl}(r) = -\left\{\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\right\}^{1/2} e^{-r/na_0}\left(\frac{2r}{na_0}\right)^l L_{n+l}^{2l+1}\left(\frac{2r}{na_0}\right)$$

and L_{n+l}^{2l+1} stands for the associated Laguerre polynomial. Using our realization (6) and (8), we have derived the commutator brackets (10) and (12). From these two commutation relations and selection rules (16) and (17), we can derive

$$\begin{aligned} &\langle n'l'm|J_+|n''l''m-1\rangle\langle n''l''m-1|J_-|nlm\rangle \\ &\quad -\langle n'l'm|J_-|n''l''m+1\rangle\langle n''l''m+1|J_+|nlm\rangle \\ &\quad = \delta_{nn'}\delta_{ll'}m\left\{\frac{4E_n}{\mu e^4}[2l(l+1)+1]+2\right\} \end{aligned} \tag{18}$$

and

$$\langle n'l'm \pm 1|J_3|n''l''m \pm 1\rangle\langle n''l''m \pm 1|J_\pm|nlm\rangle$$
$$- \langle n'l'm \pm 1|J_\pm|n''l''m\rangle\langle n''l''m|J_3|nlm\rangle$$
$$= \pm\delta_{nn'}\delta_{ll'}\sqrt{(l \mp m)(l \pm m + 1)}\left\{\frac{2E_n}{\mu e^4}[2l(l+1)+1]+1\right\} \quad (19)$$

Using the relation

$$\langle n'l'm - 1|J_-|nlm\rangle = \langle nlm|J_+|n'l'm - 1\rangle$$

one can verify that

$$\langle nlm|J_+|n'l'm - 1\rangle$$
$$= \delta_{nn'}\delta_{ll'}\sqrt{\left\{\frac{2E_n}{\mu e^4}[2l(l+1)+1]+1\right\}(l-m)(l+m+1)} \quad (20)$$

and

$$\langle nlm|J_3|n'l'm\rangle = \delta_{nn'}\delta_{ll'}m\sqrt{\left\{\frac{2E_n}{\mu e^4}[2l(l+1)+1]+1\right\}} \quad (21)$$

obey equations (18) and (19). We see that the $Y(sl(2))$ generators realized by (6) and (8) also represent the transition within the states with the same principal quantum number n and angular quantum number l. This is consistent with the general discussion in the last paragraph.

In conclusion, we have obtained that there is an algebraic structure of the Yangian $Y(sl(2))$ in the hydrogen atom. Its generators have been expressed in terms of the usual angular momentum and Runge–Lenz vector. We found that the energy plays the role of a deformation parameter in our realization of the $Y(sl(2))$ generators. It has been seen that the algebraic structure at the critical point of transition of the hydrogen atom from a bound state to a free state is different from that corresponding to the other energy value, which has the loop algebra $L(sl(2))$ [more precisely, the subalgebra of $L(sl(2))$] (Bernard, 1993) structure. The matrix elements of the $Y(sl(2))$ generators for the eigenstates of the hydrogen atom (1) have also been determined.

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